Univalent categories and the Rezk completion

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When formalizing category theory in traditional, set-theoretic foundations, a significant discrepancy between the foundational notion of "sameness"—*equality*—and its categorical notion arises: most category–theoretic concepts are invariant under weaker notions of sameness than equality, namely isomorphism in a category or equivalence of categories. We show that this discrepancy can be avoided when formalizing category theory in Univalent Foundations.

The *Univalent Foundations* is an extension of Martin-Löf Type Theory (MLTT) recently proposed by V. Voevodsky [4]. Its novelty is the *Univalence Axiom* (UA) which closes an unfortunate incompleteness of MLTT by providing "more equalities between types". This is obtained by identifying equality of types with equivalence of types. To prove two types equal, it thus suffices to construct an equivalence between them.

When formalizing category theory in the Univalent Foundations, the idea of Univalence carries over. We define a *precategory* to be given by a type of objects and, for each pair (x, y) of objects, a *set* $\hom(x, y)$ of morphisms, together with identity and composition operations, subject to the usual axioms. In the Univalent Foundations, a type X is called a *set* if it satisfies the principle of Uniqueness of Identity Proofs, that is, for any $x, y : X$ and $p, q : \mathsf{Id}(x, y)$, the type $\mathsf{Id}(p, q)$ is inhabited. This requirement avoids the introduction of coherence axioms for associativity and unitality of categories.

A *univalent* category is then defined to be a category where the type of isomorphisms between any pair of objects is equivalent to the identity type between them. We develop the basic theory of such univalent categories: functors, natural transformations, adjunctions, equivalences, and the Yoneda lemma.

Two categories are called *equivalent* if there is a pair of adjoint functors between them for which the unit and counit are natural isomorphisms. Given two categories, one may ask whether they are equal in the type-theoretic sense—that is, if there is an identity term between them in the type of categories—or whether they are equivalent. One of our main results states that for univalent categories, the notion of (type-theoretic) *equality* and (category-theoretic) *equivalence coincide*. This implies that properties of univalent categories are automatically invariant under equivalence of categories—an important difference to the classical notion of categories in set theory, where this invariance does not hold.

Moreover, we show that any category is weakly equivalent to a univalent category—its *Rezk completion*—in a universal way. It can be considered as a truncated version of the Rezk completion for Segal spaces [3]. The Rezk completion of a category is constructed via the Yoneda embedding of a category into its presheaf category, a construction analogous to the *strictification* of bicategories by the Yoneda embedding into Cat, the 2-category of categories.

Large parts of this development have been formally verified [1] in the proof assistant Coq, building on Voevodsky's *Foundations* library [5]. In particular, the formalization includes the Rezk completion together with its universal property.

A preprint covering the content of this talk is available on the arXiv [2].

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