Displayed categories

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Outline

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1. Goals and background
2. Displayed category theory
3. Fibrations and comprehension categories
4. Univalence
5. Creation of limits

Context

- Open problem: Initiality conjecture for dependently typed theories
    1. Develop notion of 'signature' for type theories
    2. Construct initial model for any signature
- Project with Peter Lumsdaine and Vladimir Voevodsky: *Comparing categorical structures for type theory*
    - Categories with families
    - Type categories
    - Categories with display maps
    - Comprehension categories

  and formalize results in (univalent) type theory.

Displayed categories help with two challenges encountered in this project.

Goals

Displayed categories help with two challenges:

Avoid reasoning about equality of objects of categories

Equality of objects used in classical formulations of several concepts:

- (Grothendieck) fibrations
- Creation of limits

Build categories of complex structures step-wise

- Toy example: category of groups from category of sets + extra structure
- Specifically: mathematical status for extra structure

Logical setting

Type theory with different possible interpretations

naïve: types interpreted as sets

univalent: types interpreted as simplicial sets

Some issues and results trivialize in naïve interpretation

- Transport along equalities
- Results on univalent categories

Type-theoretical background

- Type theory with Σ, Π, $=$, 0, 1, 2, $\mathbf{N}$, $\cup$
- Type A is **contractible** if has a unique inhabitant
- Type A is a **proposition** if all inhabitants are equal
- Type A is a **set** if all its identity types $a = a'$ are propositions

Results do not rely on univalence or Axiom K

Formalization

- Many of the results formalized, based on the `UniMath` library
- Available on `https://github.com/UniMath/TypeTheory`
- Ca. 5000 loc

Categories

A category $\mathscr{C}$ is

- a type $\mathscr{C}_o$ of objects
- for any two objects $a, b : \mathscr{C}_o$, a **set** $a \to b$ of arrows
- for any $a : \mathscr{C}_o$, an arrow $1_a : a \to a$
- composition: $(a \to b) \times (b \to c) \longrightarrow (a \to c)$, denoted $f \cdot g$
- axioms postulating identities of arrows

Categories

A **univalent** category $\mathscr{C}$ is

- a type $\mathscr{C}_0$ of objects
- for any two objects $a, b : \mathscr{C}_0$, a **set** $a \to b$ of arrows
- for any $a : \mathscr{C}_0$, an arrow $1_a : a \to a$
- composition: $(a \to b) \times (b \to c) \longrightarrow (a \to c)$, denoted $f \cdot g$
- axioms postulating identities of arrows
- such that the map

$$\mathsf{idtoiso} : \prod_{a,b:\mathscr{C}_0} \big((a = b) \to \mathsf{Iso}(a,b)\big)$$

is an equivalence 'pointwise', i.e., for any $a, b : \mathscr{C}_0$,

$$\mathsf{idtoiso}_{a,b} : a = b \xrightarrow{\simeq} \mathsf{Iso}(a,b)$$

Outline

Displayed categories

Given a category $\mathscr{C}$, a **displayed category** $\mathscr{D}$ **over** $\mathscr{C}$ consists of

- for each $c : \mathscr{C}$, a type $\mathscr{D}_c$
- for each $f : a \to b$ of $\mathscr{C}$ and $x : \mathscr{D}_a$ and $y : \mathscr{D}_b$, a **set** $\mathsf{hom}_f(x, y)$
- for each $c : \mathscr{C}$ and $x : \mathscr{D}_c$, a morphism $1_x : \mathsf{hom}_{1_c}(x, x)$
- for all $f : a \to b$ and $g : b \to c$ in $\mathscr{C}$ and $x : \mathscr{D}_a$ and $y : \mathscr{D}_b$ and $z : \mathscr{D}_c$, a function

  $$(\cdot) : \mathsf{hom}_f(x, y) \times \mathsf{hom}_g(y, z) \to \mathsf{hom}_{f \cdot g}(x, z),$$

  denoted by $(\bar{f}, \bar{g}) \mapsto \bar{f} \cdot \bar{g} : \mathsf{hom}_{f \cdot g}(x, z)$
- **laws—well-typed modulo axioms of** $\mathscr{C}$

Total category of a displayed category

The **total category** $\int \mathscr{D}$ of $\mathscr{D}$ over $\mathscr{C}$

- objects are pairs (a,x) where $a : \mathscr{C}$ and $x : \mathscr{D}_a$
- maps $(a,x) \to (b,y)$ are pairs $(f,\bar{f})$ where $f : a \to b$ and $\bar{f} : \mathsf{hom}_f(x,y)$

Forgetful functor

$$\pi_1^{\mathscr{D}} : \int \mathscr{D} \to \mathscr{C}$$

Displayed categories over $\mathscr{C}$ are the same as 'a category and a functor into $\mathscr{C}$'.

Examples

The **category of groups** is the total category of the displayed category grp, over set:

- $\mathsf{grp}_X :=$ set of group structures on the set X
- for a function $f : X \to Y$ and group structures (μ, e) on X and (μ', e') on Y,

$$\mathsf{hom}_f((\mu, e), (\mu', e')) :=$$
$$f \text{ is a homorphism with respect to } (\mu, e), (\mu', e')$$

Similarly for **category of topological spaces**.

More examples

- Any category is displayed over **1**.
- Given a predicate $P : \mathscr{C}_0 \to \mathsf{type}$, setting $\mathscr{D}_c := P(c)$ and $\mathsf{hom}_f(x,y) = 1$ yields

$$\int \mathscr{D} = \text{ full subcategory spanned by } P$$

- If every displayed hom-set $\mathsf{hom}_f(x,y)$ of $\mathscr{D}$ is a proposition (inhabited, contractible) then $\pi_1 : \int \mathscr{D} \to \mathscr{C}$ is faithful (full, fully faithful).
- Total category of displayed (co)slice category is arrow category

$$\mathscr{C}^{\to} \;\simeq\; \int_{c:\mathscr{C}} \mathscr{C}/c \;\simeq\; \int_{c:\mathscr{C}} c\backslash\mathscr{C}$$

but the π_1's are different.

Displayed functors

Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor, and $\mathscr{D}$ over $\mathscr{C}$ and $\mathscr{D}'$ over $\mathscr{C}'$.
A **(displayed) functor** G **from** $\mathscr{D}$ **to** $\mathscr{D}'$ **over** F consists of:

- for each $c : \mathscr{C}$, a map

$$G_c : \mathscr{D}_c \to \mathscr{D}'_{Fc}$$

- for each $f : c \to c'$ in $\mathscr{C}$, a map

$$\mathsf{hom}_f(x, y) \to \mathsf{hom}_{Ff}(Gx, Gy)$$

- dependent analogues of the usual functor laws

Induces **total functor** $\int G : \int \mathscr{D} \to \int \mathscr{D}'$ commuting with the forgetful functors.

Displayed X over X in the base, inducing X of total categories

For X being

- natural transformations
- adjunctions
- equivalences

In particular,

- displayed category of displayed functors from $\mathscr{D}$ to $\mathscr{D}'$ over category of functors from $\mathscr{C}$ to $\mathscr{C}'$

Fibre categories

Given $\mathscr{D}$ over $\mathscr{C}$ and c an object of $\mathscr{C}$, define **fibre category** $\mathscr{D}_c$

- $(\mathscr{D}_c)_0 := \mathscr{D}_c$
- $\mathsf{hom}(x, y) := \mathsf{hom}_{1_c}(x, y)$

But: displayed X do not generally restrict to X on fibres, requires well-behaved displayed category $\mathscr{D}$

Outline

Fibrations

Definition (cartesian lift, classically)

Given $F : \mathscr{D} \to \mathscr{C}$ and $f : c' \to c$ in $\mathscr{C}$ and $d : \mathscr{D}_0$ such that $Fd = c$, a cartesian lift of (f, d) is an object $d' : \mathscr{D}_0$ with $Fd' = c'$ and a cartesian map $f' : d' \to d$ with $Ff' = f$.

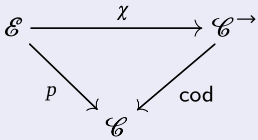
Definition (cartesian lift in terms of displayed categories)

Given $\mathscr{D}$ a displayed category over $\mathscr{C}$ and $f : c' \to c$ in $\mathscr{C}$ and $d : \mathscr{D}_c$, a cartesian lift of (f, d) is an object $d' : \mathscr{D}_{c'}$ and a cartesian map $\bar{f} : \mathsf{hom}_f(d', d)$.

A **fibration** is a displayed category with a cartesian lift for any $f : c' \to c$ and $d : \mathscr{D}_c$.

Comprehension categories

Definition (comprehension category, classically)

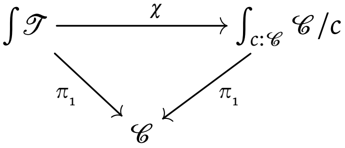


commuting **strictly**

Definition (comprehension category via displayed categories)

- a fibration (in particular, displayed category) $\mathscr{T}$ over $\mathscr{C}$
- a displayed functor $\mathscr{T} \to \mathscr{C}/-$ over identity functor on $\mathscr{C}$

Induces a strictly commuting triangle of functors



Outline

Univalent displayed categories

- Given $\mathscr{D}$ over $\mathscr{C}$ and $i : c \cong c'$ in $\mathscr{C}$, write $\mathsf{Iso}_i(d, d')$ for type of **displayed isomorphisms**
- For $e : c = c'$ and $d : \mathscr{D}_c$ and $d' : \mathscr{D}_{c'}$,

$$\mathsf{idtoiso}_{e,d,d'} : (d =_e d') \to \mathsf{Iso}_{\mathsf{idtoiso}(e)}(d, d')$$

- Call $\mathscr{D}$ **(displayedly) univalent** if $\mathsf{idtoiso}_{e,d,d'}$ is an equivalence for all e, d, d'.

Lemma

$\mathscr{D}$ displayedly univalent iff all fibre categories $\mathscr{D}_c$ univalent

Structure Identity Principle

Theorem

Given $\mathscr{D}$ over $\mathscr{C}$, if $\mathscr{C}$ is univalent and $\mathscr{D}$ is (displayedly) univalent, then $\int \mathscr{D}$ is univalent.

- Gives a modular way to show that categories of complicated structures are univalent.
- Structure Identity Principle (Aczel, Coquand & Danielsson) is a special case.

Outline

Creation of limits

Definition (classically)

A functor $F : \mathscr{A} \to \mathscr{B}$ **creates limits** of shape I if for any diagram $D : I \to \mathscr{A}$

- for any limit cone $C : \underline{B} \to FD$ on diagram FD there is a unique cone $C' : \underline{A} \to D$ such that $F(C') = C$
- C' is limit cone for D

Definition (in terms of displayed categories)

Let $\mathscr{D}$ be a displayed category over $\mathscr{C}$ and I a category. We say that $\mathscr{D}$ creates limits of shape I if ...

A displayed category $\mathscr{D}$ over a category $\mathscr{C}$ creates limits (of shape I) if and only the functor $\pi_1^{\mathscr{D}} : \int \mathscr{D} \to \mathscr{C}$ creates limits (of shape I) in the classical sense.

Creation of limits II

Lemma

Suppose the category $\mathscr{C}$ has limits of shape I, and the displayed category $\mathscr{D}$ over $\mathscr{C}$ creates limits of shape I. Then $\int \mathscr{D}$ has all such limits, and $\pi_1^{\mathscr{D}} : \int \mathscr{D} \to \mathscr{C}$ preserves them.

Examples

- Given $F : \mathscr{C} \to \mathscr{C}$, the displayed category of F-algebras over $\mathscr{C}$ creates limits. Same for monad algebras.
- The displayed category of groups over sets creates limits.

Future work

- Develop notion of *displayed limit* encompassing and generalizing the creation of limits
- Assemble displayed categories into a *displayed bicategory over the bicategory of categories*
- Displayed categories form a sort of 2-dimensional 'category with display maps', with displayed categories over $\mathscr{C}$ being the 'types in context $\mathscr{C}$' $\rightsquigarrow$ directed type theory

Future work

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Thanks for your attention!