Categorical structures for type theory in univalent foundations

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Outline

Overall goal

Compare various categorical structures arising in study of type theory:

- C-systems
- Categories with Families
- (Split) type categories
- Categories with Display Maps
- . . .

by constructing maps between them and proving properties of those maps.

What are those categorical structures?

- Gadgets are used to interpret syntax
- Additional structure on those gadgets is usually considered (Π, Σ, Id,. . .)

Disclaimer:

- This talk is not about syntax nor about interpretation of syntax.
- In this talk additional structure is not considered.

How to compare?

- In set theory and plain type theory, comparison on the level of categories—i.e., up to **isomorphism**
- In **univalent** type theory, can meaningfully compare on level of types—i.e., up to **identity**

In this talk

- We consider three kinds of structures
  - Categories with Families
  - Split type categories
  - Representable maps of presheaves
- We compare them in univalent type theory
- Results are formalized in UniMath

The type theory we work in

- Identity types
- Sigma types
- Dependent function types
- Some base types (natural numbers, booleans, . . .)
- Disjoint sum types
- Propositional truncation
- Univalence axiom
- Resizing rule (not used in this work)

For more details, see my talk on UniMath.

Questions studied in this talk

1. Compare different structures over same category $\mathscr{C}$, e.g.,

$$\mathsf{cwf}(\mathscr{C}) \xrightarrow{?} \mathsf{splty}(\mathscr{C})$$

2. Transfer of one structure along a functor $F : \mathscr{C} \to \mathscr{D}$, e.g.,

$$\mathsf{cwf}(\mathscr{C}) \xrightarrow{?} \mathsf{cwf}(\mathscr{D})$$

3. In particular, transfer of a structure on $\mathscr{C}$ to its Rezk completion, e.g.,

$$\mathsf{cwf}(\mathscr{C}) \xrightarrow{?} \mathsf{cwf}(\mathsf{RC}(\mathscr{C}))$$

Constructions presented in this talk

$$
\begin{array}{ccccc}
\mathsf{splty}(\mathscr{C}) & \xleftrightarrow{\simeq} & \mathsf{cwf}(\mathscr{C}) & \longrightarrow & \mathsf{cwf}(\mathsf{RC}(\mathscr{C})) \\
 & & \downarrow & & \updownarrow \simeq \\
 & & \mathsf{rep}(\mathscr{C}) & \xleftrightarrow{\simeq} & \mathsf{rep}(\mathsf{RC}(\mathscr{C}))
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$$
\begin{array}{ccccccccc}
\mathsf{splty}(\mathscr{C}) & \xleftrightarrow{\simeq} & \mathsf{cwf}(\mathscr{C}) & \xleftrightarrow{\simeq} & \mathsf{relu}(\mathsf{y}_{\mathscr{C}}) & \longrightarrow & \mathsf{relu}(\mathsf{y}_{\mathsf{RC}(\mathscr{C})}) & \xleftrightarrow{\simeq} & \mathsf{cwf}(\mathsf{RC}(\mathscr{C})) \\
 & & \big\downarrow & & \big\downarrow & & \big\updownarrow \simeq & & \big\updownarrow \simeq \\
 & & \mathsf{rep}(\mathscr{C}) & \xleftrightarrow{\simeq} & \mathsf{relwku}(\mathsf{y}_{\mathscr{C}}) & \xleftrightarrow{\simeq} & \mathsf{relwku}(\mathsf{y}_{\mathsf{RC}(\mathscr{C})}) & \xleftrightarrow{\simeq} & \mathsf{rep}(\mathsf{RC}(\mathscr{C}))
\end{array}
$$

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Univalent categories

A category is

- a type $O : \mathsf{U}$ of objects
- a dependent type $A : O \times O \to \mathsf{Set}$ of arrows
- for any $a : O$, an arrow $1_a : A(a, a)$
- composition
- axioms postulating identities of arrows

Univalent categories

A **univalent** category is

- a type $O : \mathsf{U}$ of objects
- a dependent type $A : O \times O \to \mathsf{Set}$ of arrows
- for any $a : O$, an arrow $1_a : A(a, a)$
- composition
- axioms postulating identities of arrows
- such that the map

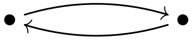
$$\mathsf{idtoiso} : \prod_{a,b:O} (a = b) \to \mathsf{Iso}(a, b)$$

  is an equivalence for any a, b

Examples of univalent categories

- Set
- Groups, rings, . . . (Structure Identity Principle)
- Functor category $[\mathscr{C}, \mathscr{D}]$, if $\mathscr{D}$ is univalent
- Full subcategories of univalent categories

Non-example:

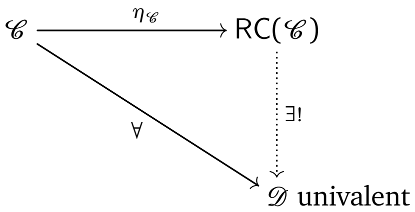


More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, then the category with objects X and $\hom(x,y) := (x = y)$ is univalent
- If $\mathscr{C}$ is univalent, then the category of cones of shape $F : \mathscr{J} \to \mathscr{C}$ is univalent
  - ⇝ limits (limiting cones) in a univalent category are unique **up to identity**

Rezk completion

To any category $\mathscr{C}$, associate its 'Rezk completion' $\mathsf{RC}(\mathscr{C})$,



- $\mathsf{RC}(\mathscr{C})$ is univalent
- $\eta_{\mathscr{C}}$ is a weak equivalence (fully faithful and essentially surjective)

Outline

Categories with Families

A *category with families (à la Fiore)* consists of:

0. a category $\mathscr{C}$, together with
1. presheaves $\mathsf{Ty}, \mathsf{Tm} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$;
2. a natural transformation $\mathsf{p} : \mathsf{Tm} \to \mathsf{Ty}$; and
3. for each object $\Gamma : \mathscr{C}$ and $A : \mathsf{Ty}(\Gamma)$,
    1. an object $\Gamma.A : \mathscr{C}$ and map $\pi_A : \Gamma.A \to \Gamma$,
    2. an element $\mathsf{te}_A : \mathsf{Tm}(\Gamma.A)$, such that $\mathsf{p}_{\Gamma.A}(\mathsf{te}_A) = \pi_A^* A$
    3. and such that the induced commutative square

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \xrightarrow{\widehat{\mathsf{te}_A}} & \mathsf{Tm} \\ {\scriptstyle \mathsf{y}(\pi_A)}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{p}} \\ \mathsf{y}(\Gamma) & \xrightarrow[\widehat{A}]{} & \mathsf{Ty} \end{array}$$

is a pullback.

Type categories

A *type-category* consists of:

0. a category $\mathscr{C}$, together with
1. for each object $\Gamma : \mathscr{C}$, a type $\mathsf{Ty}(\Gamma)$,
2. for each $\Gamma : \mathscr{C}$ and $A : \mathsf{Ty}(\Gamma)$, an object $\Gamma.A : \mathscr{C}$
3. for each such Γ, A, a morphism $\pi_A : \Gamma.A \to \Gamma$,
4. for each map $f : \Gamma' \to \Gamma$, a ‘reindexing’ function $\mathsf{Ty}(\Gamma) \to \mathsf{Ty}(\Gamma')$, denoted $A \mapsto f^*A$,
5. for each Γ, $A : \mathsf{Ty}(\Gamma)$, and $f : \Gamma' \to \Gamma$, a morphism $\mathsf{q}(f, A) : \Gamma'.f^*A \to \Gamma.A$,
6. such that for each such Γ, A, Γ', f, the following square commutes and is a pullback:

$$\begin{array}{ccc} \Gamma'.f^*A & \xrightarrow{\mathsf{q}(f,A)} & \Gamma.A \\ {\scriptstyle \pi_{f^*A}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_A} \\ \Gamma' & \xrightarrow[f]{} & \Gamma \end{array}$$

Split type categories

A type-category is *split* if:

1. for each Γ, the type $\mathsf{Ty}(\Gamma)$ is a set;
2. for each Γ and $A : \mathsf{Ty}(\Gamma)$, we have equalities
   1. $e : 1_\Gamma^* A = A$, and
   2. $\mathsf{q}(1_\Gamma, A) = \Delta_e \ : \ \Gamma.1_\Gamma^* A \to \Gamma.A$; and
3. for $f' : \Gamma'' \to \Gamma')$, $f : \Gamma' \to \Gamma$, and $A : \mathsf{Ty}(\Gamma)$, we have equalities
   1. $e' : (f' \cdot f)^* A = f'^* f^* A$, and
   2. $\mathsf{q}(f' \cdot f, A) = \Delta_{e'} \cdot \mathsf{q}(f', f^* A) \cdot \mathsf{q}(f, A) \ : \ \Gamma''.(f' \cdot f)^* A \to \Gamma.A$.

Goal

For a given category $\mathscr{C}$,

$$\mathsf{splty}(\mathscr{C}) \simeq \mathsf{cwf}(\mathscr{C})$$

Reducing complexity: factoring out

A *(split) object extension structure* on a category $\mathscr{C}$ consists of:

1. a functor $\mathsf{Ty} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$;
2. for each $\Gamma : \mathscr{C}$ and $A : \mathsf{Ty}(\Gamma)$,
   1. an object $\Gamma.A$, and
   2. a *projection* morphism $\pi_A : \mathscr{C}(\Gamma.A, \Gamma)$.

Trivial equivalences

- A cwf structure on $\mathscr{C}$ is a pair of an object extension structure X on $\mathscr{C}$ and a **term structure on** X.
- A splty structure on $\mathscr{C}$ is a pair of an object extension structure X on $\mathscr{C}$ and a q**-morphisms structure on** X.

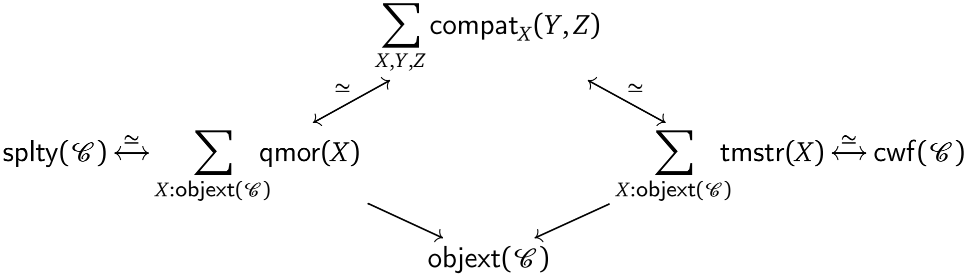
Reduced goal

Given an object extension structure X on a category $\mathscr{C}$, construct

$$\mathsf{qmor}(X) \simeq \mathsf{tmstr}(X)$$

- Terms are defined as sections to canonical projections.
- Showing that this is an equivalence of types requires function extensionality in one direction, and univalence in the other.

Equivalence between splty's and cwf's



Outline

Goal

Given $F : \mathscr{C} \to \mathscr{D}$ a suitably 'good' functor, construct

$$\mathsf{cwf}(\mathscr{C}) \to \mathsf{cwf}(\mathscr{D})$$

Special case:

$$\mathsf{cwf}(\mathscr{C}) \to \mathsf{cwf}(\mathsf{RC}(\mathscr{C}))$$

Conveniently studied at level of relative universes.

Universe categories

A **universe** in a category $\mathscr{D}$ is

- a morphism $\mathsf{p} : \tilde{\mathsf{U}} \to \mathsf{U}$ in $\mathscr{D}$ and
- a choice of pullback for any $f : X \to U$,

$$
\begin{array}{ccc}
X' & \xrightarrow{Q} & \tilde{\mathsf{U}} \\
{\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{p}} \\
X & \xrightarrow[f]{} & \mathsf{U}
\end{array}
$$

A **universe relative to** $J : \mathscr{C} \to \mathscr{D}$ is

- a morphism $\mathsf{p} : \tilde{\mathsf{U}} \to \mathsf{U}$ in $\mathscr{D}$ and
- a choice of 'J-pullback' for any $f : J(X) \to U$,

$$
\begin{array}{ccc}
J(X') & \xrightarrow{Q} & \tilde{\mathsf{U}} \\
{\scriptstyle J(p')}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{p}} \\
J(X) & \xrightarrow[f]{} & \mathsf{U}
\end{array}
$$

Relative universes and cwf's

Denote

$$\mathsf{y}_{\mathscr{C}} : \mathscr{C} \to \mathsf{PreShv}(\mathscr{C})$$

the Yoneda functor. Then

$$\mathsf{cwf}(\mathscr{C}) \simeq \mathsf{relu}(\mathsf{y}_{\mathscr{C}})$$

Transfer of relative universe structures

$$
\begin{array}{ccc}
\mathscr{C} & \xrightarrow{\;J\;} & \mathscr{D} \\
{\scriptstyle R}\downarrow & \swArrow_{\alpha} & \downarrow{\scriptstyle S} \\
\mathscr{C}' & \xrightarrow[\;J'\;]{} & \mathscr{D}',
\end{array}
$$

- S preserves pullbacks
- S is full
- R is essentially surjective
- $\mathscr{C}'$ is univalent
- J' is fully faithful

Given a J-relative universe structure on a map $\mathsf{p} : \tilde{\mathsf{U}} \to \mathsf{U}$ in $\mathscr{D}$, construct a J'-universe structure on $S(\mathsf{p})$ in $\mathscr{D}'$.

Transfer of cwf structures

Given $F : \mathscr{C} \to \mathscr{D}$ a weak equivalence and $\mathscr{D}$ univalent, instantiating

$$\begin{array}{ccc}
\mathscr{C} & \xrightarrow{\mathsf{y}_{\mathscr{C}}} & \mathsf{PreShv}(\mathscr{C}) \\
{\scriptstyle F}\big\downarrow & \swarrow_{\alpha} & {\scriptstyle F^\circ}\big\uparrow \simeq \big\downarrow{\scriptstyle (F^\circ)^{-1}} \\
\mathscr{D} & \xrightarrow[\mathsf{y}_{\mathscr{D}}]{} & \mathsf{PreShv}(\mathscr{D})
\end{array}$$

yields

$$\mathsf{cwf}(\mathscr{C}) \to \mathsf{cwf}(\mathscr{D})$$

and, in particular

$$\mathsf{cwf}(\mathscr{C}) \to \mathsf{cwf}(\mathsf{RC}(\mathscr{C}))$$

Outline

Representable maps of presheaves

- Similar to category with families, but where the pullbacks are not chosen, but only exist.
- Forget the choice of pullbacks:

$$\mathsf{cwf}(\mathscr{C}) \to \mathsf{rep}(\mathscr{C})$$

- Constructing

$$\mathsf{rep}(\mathscr{C}) \to \mathsf{cwf}(\mathscr{C})$$

  would require AC, in general.

Categories with families and representable maps

If $\mathscr{C}$ is univalent,

$$\mathsf{cwf}(\mathscr{C}) \simeq \mathsf{rep}(\mathscr{C})$$

Uses that the choice of structure is unique up to identity (not just up to isomorphism) in $\mathscr{C}$.

Summary

$$
\begin{array}{ccccc}
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$$

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Thanks for your attention!

References

- First occurrence of univalent categories:
  Hofmann & Streicher, *The groupoid interpretation of type theory*
- Categories with families:
  Dybjer, *Internal type theory*, TYPES'95
  Fiore, *Discrete Generalised Polynomial Functors*, Talk at ICALP 2012 (Appendix) `http://www.cl.cam.ac.uk/~mpf23/talks/ICALP2012.pdf`
- (Split) type categories:
  Pitts, *Categorical Logic*
  van den Berg & Garner, *Topological and simplicial models*
- Universe categories:
  Voevodsky, *A C-system defined by a universe category*

Categorical equivalence

Also construct a categorical equivalence

$$\mathsf{cwf}(\mathscr{C}) \simeq \mathsf{splty}(\mathscr{C})$$

- does not require univalence axiom
- makes use of notion of **displayed categories** to work fiberwise over a fixed category of object extension structures

Type-theoretic and categorical equivalence

Lemma

If $F : \mathscr{C} \simeq \mathscr{D}$ is an equivalence and $\mathscr{C}$ and $\mathscr{D}$ are univalent, then $F_0 : \mathscr{C}_0 \simeq \mathscr{D}_0$.

- Recover equivalence of types $\mathsf{tmstr}(X) \simeq \mathsf{qmor}(X)$ by showing that $\mathsf{tmstr}(X)$ and $\mathsf{qmor}(X)$ are univalent categories (requires univalence axiom)
- $\mathsf{cwf}(\mathscr{C})$ not in general univalent unless $\mathscr{C}$ is