Towards a Higher Structure Identity Principle

Benedikt Ahrens

with Paige R. North, Michael Shulman, Dimitris Tsementzis

work in progress

Indiscernability of identicals

Indiscernability of identicals

$$x = y \quad \rightarrow \quad \forall P\,(P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

The equivalence principle

Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

Notion of sameness depends on the objects under consideration:

- **equal** numbers, functions,...
- **isomorphic** sets, groups, rings,...
- **equivalent** categories
- **biequivalent** bicategories
- ...

Violating the equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

Violating the equivalence principle

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Exercise

Find a statement about categories that is not invariant under the equivalence of categories

A solution

"The category $\mathscr{C}$ has exactly one object."

Maybe this statement is simply silly!

A language for invariant properties

Michael Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a* ***constraint on the language*** *of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

A language for invariant properties

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Makkai's goal

to devise a language in which only invariant properties can be expressed

$\rightsquigarrow$ First Order Logic with Dependent Sorts (FOLDS)

FOLDS is quite similar to dependent type theory

Univalent foundations and the equivalence principle

Goal

- Univalent foundations as an “invariant language”
- Invariance not only for statements, but also for constructions

In this talk

- Introduction to univalent foundations
- Invariance for set-level structures (Structure Identity Principle)
- Invariance for **univalent** categories
- Our goal: generalizing “univalence” condition to other (higher-categorical) structures and prove invariance for univalent such structures

Outline

1. Dependent type theory
2. Some important concepts in univalent foundations
3. The Structure Identity Principle for set-level structures
4. Category theory in univalent foundations
5. FOLDS categories
6. FOLDS structures

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Overview of type theory

(Dependent) Type theory

- Is a (functional programming) language of types and terms
- Has infrastructure to write mathematical statements and proofs

Write $a : A$ to say that term a has type A (e.g., $1 : \mathsf{Nat}$)

Writing well-typed programs

In type theory, both the activities of

- implementing an algorithm
- proving a mathematical statement

are done by writing well-typed programs.

Dependent types and functions

Examples of dependent types

- $\mathsf{Vect}(A, n)$ — type of vectors of length $n : \mathsf{Nat}$ of elements in type A
- $\mathsf{GrpStr}(X)$ — type of group structures on a set X

Examples of dependent functions

$$\mathsf{zero} : \prod_{m:\mathsf{Nat}} \mathsf{Vect}(\mathsf{Nat}, m)$$

$$\mathsf{tail} : \prod_{n:\mathsf{Nat}} \mathsf{Vect}(A, 1+n) \to \mathsf{Vect}(A, n)$$

$$\mathsf{GrpStrTransfer} : \prod_{X,Y:\mathsf{Set}} (X \cong Y) \times \mathsf{GrpStr}(X) \to \mathsf{GrpStr}(Y)$$

Overview of types in type theory

Type former	Notation	(special case)	canonical term
Inhabitant	$a : A$		
Dependent type	$x : A \vdash B(x)$		
Sigma type	$\sum_{x:A} B(x)$	$A \times B$	(a, b)
Product type	$\prod_{x:A} B(x)$	$A \to B$	$\lambda(x : A).b$
Coproduct type	$A + B$		$\mathsf{inl}(a)$, $\mathsf{inr}(b)$
Identity type	$a \rightsquigarrow_A b$		$\mathsf{refl}(a) : a \rightsquigarrow a$
Universe	Type		
Base types	Nat, Bool, 1, 0		

Some terms that can be defined

Induction principle for $a \rightsquigarrow b$

To define a function

$$f : \prod_{(x,y:A)} \prod_{(p:x \rightsquigarrow y)} C(x,y,p)$$

it suffices to specify its image on $(x, x, \mathsf{refl}(x))$.

- $\mathsf{false} \rightsquigarrow_{\mathsf{Bool}} \mathsf{false}$
- $\mathsf{sym} : \prod_{x,y:A} (x \rightsquigarrow y) \to (y \rightsquigarrow x)$
- $\mathsf{trans} : \prod_{x,y,z:A} (x \rightsquigarrow y) \times (y \rightsquigarrow z) \to (x \rightsquigarrow z)$

Transport

$$\mathsf{transport}^A : \prod_{x,y:A} (x \rightsquigarrow y) \to \prod_{B:A \to \mathsf{Type}} \big(B(x) \leftrightarrow B(y)\big)$$

Identities vs equalities

Inhabitants of $a \rightsquigarrow a'$ behave like equality in many ways

- reflexivity, symmetry, transitivity
- transport

Inhabitants of $a \rightsquigarrow a'$ behave **un**like equality

- Can iterate identity type
- Cannot show that any two identities are identical

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The important features of univalent foundations

Homotopy levels

- Stratification of types according to “complexity” of their identity types
- Logic: notion of propositions given by one layer of this hierarchy

Univalence axiom

Specifies the previously unspecified identity type of a universe

Logic in univalent foundations

Propositions

Type A is a proposition when we can define a term of type

$$\mathsf{isProp}(A) :\equiv \prod_{x,y:A} x \rightsquigarrow y$$

$$\mathsf{Prop} :\equiv \sum_{X:\mathsf{Type}} \mathsf{isProp}(X) \qquad \text{world of propositions}$$

- implication $A \to B$
- conjunction $A \times B$
- predicate $B : A \to \mathsf{Prop}$
- universal quantification $\prod_{a:A} B(a)$
- disjunction and existential need extra care

Contractible types, propositions and sets

- A is **contractible**

$$\mathsf{isContr}(A) :\equiv \sum_{x:A} \prod_{y:A} y \rightsquigarrow x$$

- A is a **proposition**

$$\mathsf{isProp}(A) :\equiv \prod_{x,y:A} x \rightsquigarrow y$$

- A is a **set**

$$\mathsf{isSet}(A) :\equiv \prod_{x,y:A} \mathsf{isProp}(x \rightsquigarrow y)$$

$$\mathsf{Prop} :\equiv \sum_{X:\mathsf{Type}} \mathsf{isProp}(X) \qquad \mathsf{Set} :\equiv \sum_{X:\mathsf{Type}} \mathsf{isSet}(X)$$

Contractible types, propositions and sets

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$$\mathsf{Prop} \;:\equiv\; \sum_{X:\mathsf{Type}} \mathsf{isProp}(X) \qquad \mathsf{Set} \;:\equiv\; \sum_{X:\mathsf{Type}} \mathsf{isSet}(X)$$

Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \quad :\equiv \quad \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) \rightsquigarrow b\right)$$

The type of equivalences:

$$A \simeq B \quad :\equiv \quad \sum_{f:A\to B} \mathsf{isequiv}(f)$$

Transport revisited

$$\mathsf{transport}^A : \prod_{x,y:A} (x \rightsquigarrow y) \to \prod_{B:A\to \mathsf{Type}} \big(B(x) \simeq B(y)\big)$$

The path type of pairs

Can construct equivalences

- for $s, t : A \times B$

$$\left(s \rightsquigarrow t\right) \simeq \left(\left(\mathsf{fst}(s) \rightsquigarrow \mathsf{fst}(t)\right) \times \left(\mathsf{snd}(s) \rightsquigarrow \mathsf{snd}(t)\right)\right)$$

- for $s, t : \sum_{x:A} B(x)$

$$\left(s \rightsquigarrow t\right) \simeq \left(\sum_{e:\mathsf{fst}(s) \rightsquigarrow \mathsf{fst}(t)} \mathsf{transport}^{B}\left(e, \mathsf{snd}(s)\right) \rightsquigarrow \mathsf{snd}(t)\right)$$

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Can other path types be characterized similarly?

Axioms to characterize some path types

Axiom of function extensionality for $f, g : A \to B$

$$\left(f \rightsquigarrow g\right) \simeq \left(\prod_{a:A} f(a) \rightsquigarrow g(a)\right)$$

Univalence axiom for $A, B : \mathsf{Type}$

$$\left(A \rightsquigarrow B\right) \simeq \left(A \simeq B\right)$$

More precisely: canonical map $\to$ is an equivalence.

Theorem (Voevodsky)

Univalence axiom implies function extensionality

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Transport along isomorphism

Have:

$$\mathsf{transport}_{x,y} : (x \rightsquigarrow y) \to \prod_{B:A\to\mathsf{Type}} \big(B(x) \simeq B(y)\big)$$

Want:

$$\mathsf{transport}_{x,y} : (x \cong y) \to \prod_{B:A\to\mathsf{Type}} \big(B(x) \simeq B(y)\big)$$

Suffices:

$$(x \rightsquigarrow y) \;\simeq\; (x \cong y)$$

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Monoids in type theory

In type theory, a monoid is a tuple $(M, \mu, e, \alpha, \lambda, \rho)$ where

1. $M : \mathsf{Set}$
2. $\mu : M \times M \to M$
3. $e : M$
4. $\alpha : \Pi_{(a,b,c:M)} \mu(\mu(a,b),c) \rightsquigarrow \mu(a,\mu(b,c))$
5. $\lambda : \Pi_{(a:M)} \mu(e,a) \rightsquigarrow a$
6. $\rho : \Pi_{(a:M)} \mu(a,e) \rightsquigarrow a$

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Why $M : \mathsf{Set}$?

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Why $M : \mathsf{Set}$?

Abstractly, a monoid is a (dependent) pair $(data, proof)$ where

- $data$ is 1.–3.
- $proof$ is 4.–6.

The type of monoids

- We want two monoids $(data, proof)$ and $(data', proof')$ to be the same if $data$ is the same as $data'$.
- This is guaranteed when the types of $proof$ and $proof'$ are **propositions**.
- This in turn guaranteed when the underlying type M is a **set**.

Summarily:

$$\mathsf{Monoid} :\equiv \sum_{(M:\mathsf{Set})} \sum_{(\mu,e):\mathsf{MonoidStr}(M)} \mathsf{MonoidAxioms}(M,(\mu,e))$$

Can show

$$\mathsf{isProp}(\mathsf{MonoidAxioms}(M,(\mu,e)))$$

Monoid isomorphisms

Given $\mathbf{M} \equiv (M, \mu, e, \alpha, \lambda, \rho)$ and $\mathbf{M}' \equiv (M', \mu', e', \alpha', \lambda', \rho')$, a **monoid isomorphism** is a bijection $f : M \cong M'$ preserving μ and e.

Monoid isomorphisms

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$$
\begin{aligned}
\mathbf{M} \rightsquigarrow \mathbf{M}' &\simeq (M, \mu, e) \rightsquigarrow (M', \mu', e') \\
&\simeq \sum_{p : M \rightsquigarrow M'} (\mathsf{transport}^{Y \mapsto (Y \times Y \to Y)}(p, \mu) \rightsquigarrow \mu') \\
&\qquad \times (\mathsf{transport}^{Y \mapsto Y}(p, e) \rightsquigarrow e') \\
&\simeq \sum_{f : M \cong M'} \left(f \circ m \circ (f^{-1} \times f^{-1}) \rightsquigarrow m'\right) \\
&\qquad \times (f \circ e \rightsquigarrow e') \\
&\simeq \mathbf{M} \cong \mathbf{M}'
\end{aligned}
$$

Transport along monoid isomorphism

We now have two ingredients:

1. $$\mathsf{transport}_{\mathbf{M},\mathbf{M}'} : (\mathbf{M} \rightsquigarrow \mathbf{M}') \to \prod_{B:\mathsf{Monoid}\to\mathsf{Type}} \left(B(\mathbf{M}) \simeq B(\mathbf{M}')\right)$$

2. $$(\mathbf{M} \rightsquigarrow \mathbf{M}') \;\simeq\; (\mathbf{M} \cong \mathbf{M}')$$

Composing these, we get

$$\mathsf{transport}_{\mathbf{M},\mathbf{M}'} : (\mathbf{M} \cong \mathbf{M}') \to \prod_{B:\mathsf{Monoid}\to\mathsf{Type}} \left(B(\mathbf{M}) \simeq B(\mathbf{M}')\right)$$

I.e., any structure on monoids that can be expressed in univalent type theory can be transported along isomorphism of monoids.

Structure Identity Principle

Structure Identity Principle (Aczel, Coquand&Danielsson)

For many set-level structures in univalent foundations, paths are isomorphisms.

Examples include:

- monoids, groups, rings
- posets
- discrete fields
- sets with fixpoint operator

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What about **categories**?

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EP for categories

Conjecture

For categories $\mathscr{C}$ and $\mathscr{D}$, the canonical map

$$(\mathscr{C} \rightsquigarrow \mathscr{D}) \;\to\; \mathsf{AdjEquiv}(\mathscr{C}, \mathscr{D})$$

is an equivalence.

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Counter-example

Categories in univalent foundations

Definition

A **category** $\mathscr{C}$ is given by

- a **type** $\mathscr{C}_0$: Type of **objects**
- for any $a, b : \mathscr{C}_0$, a **set** $\mathscr{C}(a,b)$: Set of **morphisms**
- operations: identity & composition

$$1_a : \mathscr{C}(a,a)$$
$$(\circ)_{a,b,c} : \mathscr{C}(b,c) \times \mathscr{C}(a,b) \to \mathscr{C}(a,c)$$

- axioms: unitality & associativity

$$1 \circ f \rightsquigarrow f \qquad f \circ 1 \rightsquigarrow f \qquad (h \circ g) \circ f \rightsquigarrow h \circ (g \circ f)$$

From paths to isomorphisms

Definition (univalent category)

For a category $\mathscr{C}$ we define

$$\mathsf{idtoiso} : \prod_{a,b:\mathscr{C}_0} (a \rightsquigarrow b) \to \mathsf{iso}(a,b)$$

$$\mathsf{idtoiso}(a,a,\mathsf{refl}(a)) \quad :\equiv \quad 1_a$$

We call the category $\mathscr{C}$ **univalent** if, for any objects $a, b : \mathscr{C}_0$,

$$\mathsf{idtoiso}_{a,b} : (a \rightsquigarrow b) \to \mathsf{iso}(a,b)$$

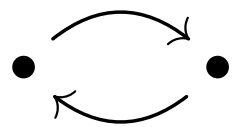
is an equivalence of types.

Examples of univalent categories

- Set
- Groups, rings, . . . (Structure Identity Principle)
- Functor category $[\mathscr{C}, \mathscr{D}]$, if $\mathscr{D}$ is univalent
- Full subcategories of univalent categories
- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, then there is a univalent category with X as objects and $\hom(x,y) \;:\equiv\; (x \rightsquigarrow y)$
- If $\mathscr{C}$ is univalent, then the category of cones of shape $F : \mathscr{J} \to \mathscr{C}$ is
  - $\rightsquigarrow$ limits (limiting cones) in a univalent category are unique **up to paths**

Non-univalent categories

- Any "chaotic" category $\mathscr{C}$ with $\mathscr{C}(x,y) \simeq 1$, for $\mathscr{C}_0$ not a proposition



- Any chaotic category $\mathscr{C}$ with an object $c : \mathscr{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$
  - $\rightsquigarrow$ a category can be equivalent to a univalent one without being univalent itself

(Adjoint) equivalence of categories

An **equivalence** $\mathscr{C} \simeq \mathscr{D}$ is given by

- a functor $F : \mathscr{C} \to \mathscr{D}$
- a functor $G : \mathscr{D} \to \mathscr{C}$
- a natural isomorphism $\eta : 1_{\mathscr{C}} \xrightarrow{\cong} GF$
- a natural isomorphism $\epsilon : FG \xrightarrow{\cong} 1_{\mathscr{D}}$

(F, G, η, ϵ) is an adjoint equivalence if F and G form an adjunction.

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Theorem

*For **univalent** categories $\mathscr{C}$ and $\mathscr{D}$, the canonical map*

$$(\mathscr{C} \rightsquigarrow \mathscr{D}) \;\to\; \mathsf{AdjEquiv}(\mathscr{C}, \mathscr{D})$$

is an equivalence.

Goal

Prove a similar theorem for other structures.

Envisioned result

Given a signature $\mathscr{L}$, and two $\mathscr{L}$-univalent $\mathscr{L}$-structures M and N, then

$$(M \rightsquigarrow N) \xrightarrow{\simeq} (M \simeq_{\mathscr{L}} N)$$

Need notions of

- signature
- $\mathscr{L}$-structure for a signature $\mathscr{L}$
- $\mathscr{L}$-equivalence of $\mathscr{L}$-structures
- $\mathscr{L}$-isomorphism and $\mathscr{L}$-univalence

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Before, reformulate the example of categories.

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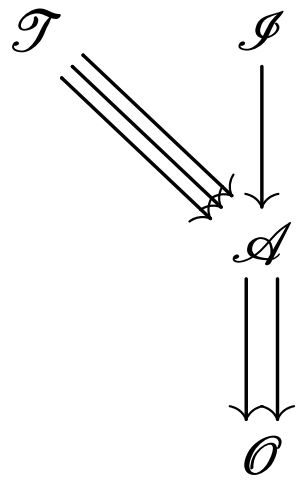
Makkai's FOLDS

- Signature := inverse category

$\mathscr{L}_{\text{cat}} :=$ 

+ equalities of arrows

- Structures for a signature $\mathscr{L} := \mathscr{L} \to \mathsf{Set}$
- Signature only specifies the data, e.g., of categories
- FOLDS language for axioms

FOLDS structures in type theory

In type theory natural to define a structure for $\mathscr{L}_{\text{cat}}$ to be

- O : Type;
- $A : O \times O \to \text{Type}$;
- $I : \prod_{x:O} A(x,x) \to \text{Type}$; and
- $T : \prod_{x,y,z:O} A(x,y) \to A(y,z) \to A(x,z) \to \text{Type}$.

Here I, T Type-valued predicates "being an identity", "being a composite"

Attention

Axioms not part of $\mathscr{L}_{\text{cat}}$-structures, have to be imposed additionally:

1. Existence and uniqueness of composition
2. Existence of identity arrows
3. Categorical axioms

Comparison 1-univalent FOLDS-cats vs categories

Definition (1-univalent FOLDS-cat)

An $\mathscr{L}_{\text{cat}}$-structure such that

- $I_x(f)$, $T_{x,y,z}(f,g,h)$, and $E_{x,y}(f,g)$ are propositions
- $A(x,y)$ is a set,
- and the axioms 1–3 are satisfied.

Equivalence of types

$$\text{1-univ. FOLDS-cats} \quad \simeq \quad \text{categories}$$

Goal

Define type of isomorphisms $a \cong_{\mathscr{L}_{\text{cat}}} b$ for 1-univ. FOLDS-cats **without referring to the axioms** 1–3

FOLDS isomorphism for categories

Goal

Define type of isomorphisms $a \cong_{\mathscr{L}_{\text{cat}}} b$ for 1-univ. FOLDS-cats **without referring to the axioms** 1–3

Using Yoneda:

- For each $x : O$, an isomorphism $\phi_{x\bullet} : A(x,a) \cong A(x,b)$.
- For each $x,y : O$, $f : A(x,y)$, $g : A(y,a)$, and $h : A(x,a)$, we have

$$T_{x,y,a}(f,g,h) \Leftrightarrow T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

(means $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$)

Problem: not symmetric, not algorithmic

FOLDS isomorphism for categories

Equivalences between hom-types with a and b substituted into *all* possible "collections of holes":

1. For each $x : O$, an equivalence $\phi_{x\bullet} : A(x,a) \simeq A(x,b)$.
2. For each $z : O$, an equivalence $\phi_{\bullet z} : A(a,z) \simeq A(b,z)$.
3. An isomorphism $\phi_{\bullet\bullet} : A(a,a) \simeq A(b,b)$.

and equivalences between all "relations with holes":

$$
\begin{aligned}
T_{x,y,a}(f,g,h) &\simeq T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h)) \\
T_{x,a,z}(f,g,h) &\simeq T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h) \\
T_{a,z,w}(f,g,h) &\simeq T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h)) \\
T_{x,a,a}(f,g,h) &\simeq T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h)) \\
T_{a,x,a}(f,g,h) &\simeq T_{b,x,b}(\phi_{\bullet x}(f), \phi_{x\bullet}(g), \phi_{\bullet\bullet}(h)) \\
T_{a,a,x}(f,g,h) &\simeq T_{b,b,x}(\phi_{\bullet\bullet}(f), \phi_{\bullet x}(g), \phi_{\bullet x}(h)) \\
T_{a,a,a}(f,g,h) &\simeq T_{b,b,b}(\phi_{\bullet\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{\bullet\bullet}(h)) \\
I_{a,a}(f) &\simeq I_{b,b}(\phi_{\bullet\bullet}(f))
\end{aligned}
$$

Univalent FOLDS categories

Theorem

In any 1-univ. FOLDS cat, the type of FOLDS-isomorphisms $a \cong_{\mathscr{L}_{cat}} b$ *is equivalent to the type of ordinary isomorphisms* $a \cong b$.

Definition

A 1-univ. FOLDS cat **is o-univalent** if for all $a, b : O$,

$$(a \rightsquigarrow b) \;\to\; (a \cong_{\mathscr{L}_{\text{cat}}} b) \text{ FOLDS-isomorphisms}$$

is an equivalence.

Theorem

A 1-univ. FOLDS cat is o-univ. if and only if its corresponding category is a univalent category.

Reminder: 1-univalent FOLDS-categories

A 1-univalent FOLDS-category consists of

- $O : \mathsf{Type}$;
- $A : O \times O \to \mathsf{Type}$;
- $I : \prod_{x:O} A(x,x) \to \mathsf{Type}$;
- $T : \prod_{x,y,z:O} A(x,y) \to A(y,z) \to A(x,z) \to \mathsf{Type}$
- $I_x(f)$ and $T_{x,y,z}(f,g,h)$ are propositions
- $A(x,y)$ is a set
- axioms 1–3

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Are implied by

- being "univalent at I and T"
- being "univalent at A"

Univalence at level A for 1-saturated FOLDS cats

Given $f, g : A(a,b)$ in a 1-saturated FOLDS cat, an isomorphism $i : f \cong_{\mathscr{L}_{\text{cat}}} g$ is given by

$$T_{x,a,b}(u,f,v) \simeq T_{x,a,b}(u,g,v)$$
$$T_{a,x,b}(u,v,f) \simeq T_{a,x,b}(u,v,g)$$
$$T_{a,b,x}(f,u,v) \simeq T_{x,a,b}(g,u,v)$$
$$\vdots$$

Observation

These types of equivalences are all propositions, and thus $(f \cong_{\mathscr{L}_{\text{cat}}} g)$ is a proposition.

Theorem

If $(f \rightsquigarrow g) \to (f \cong g)$ is an equivalence for all f, g, then $A(x,y)$ is a set for any x, y.

FOLDS categories

Definition

A **FOLDS-category** consists of the same data and axioms as a 1-saturated, but without T, I, and E being propositions and A being sets.

Have

$$(t \cong_{\mathscr{L}_{\text{cat}}} t') \simeq 1$$

since nothing "above T".

Theorem

In a FOLDS-category, T and E are propositions iff the canonical maps

$$(t = t') \to (t \cong t')$$
$$(i = i') \to (i \cong i')$$

are equivalences for all inhabitants of T and I respectively.

Conclusions from this example

- Notion of FOLDS category only requires $\mathscr{L}_{\text{cat}}$ and axioms 1–3
- Condition on homotopy levels is consequence of univalence at A, T, and I
- Notion of isomorphism and saturation conditions are independent of axioms of a FOLDS category
- (Notion of equivalence of structures does not depend on axioms)

Outline

1. Dependent type theory
2. Some important concepts in univalent foundations
3. The Structure Identity Principle for set-level structures
4. Category theory in univalent foundations
5. FOLDS categories
6. FOLDS structures

Idea: define signatures by recursion on height

- Example of categories emphasizes the significance of 'level-wise' stratification of FOLDS signatures
- Suggests defining recursively the type of signatures of height n
- Defined mutually with many other things

Signatures

- There is exactly one signature of height 0
- A signature $\mathscr{L}$ of height $n+1$ is given by
  - A signature $\mathscr{L}_{\leq n}$ of height n of 'sorts of level up to n'
  - A type $\mathscr{L}_{n+1}$ of 'sorts on level $n+1$'
  - A family of diagrams,

$$\mathscr{L} \to \mathsf{D}(\mathscr{L}_{\leq n})$$

Intuitively, $\mathsf{D}(\mathscr{L}_{\leq n})$ is "well-behaved functors $\mathscr{L}_{\leq n} \to \mathsf{Type}$".

Signatures and structures recursively

- Diagrams

$$\mathsf{D}_0(\mathscr{L}) := 1$$

$$\mathsf{D}_{n+1}(\mathscr{L}) := \sum_{M:\mathscr{L}_0 \to \mathsf{Type}} \mathsf{D}_n(\mathscr{L}'_M)$$

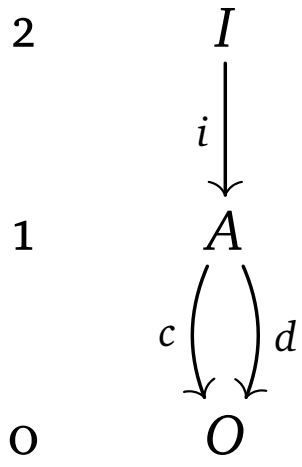
- Derivation:

$$\prod_{n:\mathsf{Nat}} \prod_{\mathscr{L}:\mathsf{Sig}(n+1)} (\mathscr{L}_0 \to \mathsf{Type}) \to \mathsf{Sig}(n)$$

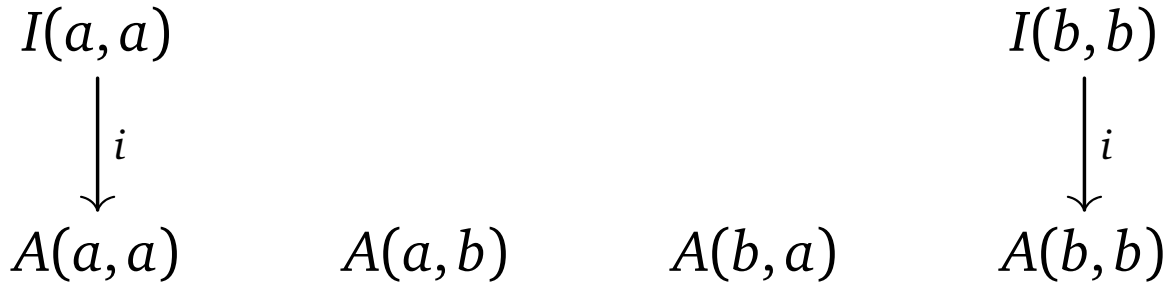
$$(\mathscr{L}, M) \mapsto \mathscr{L}'_M$$

Derivation by example

$$\prod_{n:\mathsf{Nat}} \prod_{\mathscr{L}:\mathsf{Sig}(n+1)} (\mathscr{L}_0 \to \mathsf{Type}) \to \mathsf{Sig}(n)$$



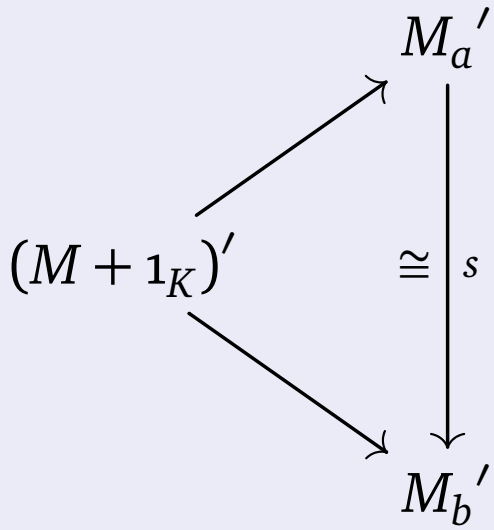
$(\mathscr{L}_{\mathrm{rg}})_0 \equiv \{O\}$
Let M_0 be a (a function picking out) the two-element set $\{a, b\}$. Then $(\mathscr{L})'_{M_0}$ is the following signature, with four sorts of rank 0 and two sorts of rank 1:



$\mathscr{L}$-isomorphism

(Pseudo-)definition

Let M be an $\mathscr{L}$-structure, $K : \mathscr{L}_0$, and $a, b : MK$. The type $a \cong_M b$ of $\mathscr{L}$-isomorphisms from a to b is defined to be the type of levelwise equivalences $s : {M_a}' \cong {M_b}'$ in $\mathsf{D}(\mathscr{L}'_{M_0+1_K})$ under $(M + 1_K)'$, i.e. those s such that the following triangle commutes:



Envisioned results

Envisioned result I

Let $\mathscr{L}$ be of height n and $K : \mathscr{L}_0$. Let M be a univalent $\mathscr{L}$-structure. Then M_0K is of h-level n.

Envisioned result II

The type of univalent structures of $\mathscr{L}$ is of h-level $h(\mathscr{L}) + 1$.

Envisioned result III

Consider $\mathscr{L}$-structures M, N for some signature $\mathscr{L}$ such that M is univalent. Then

$$(M \rightsquigarrow N) \quad \simeq \quad \text{very surjective spans between } M \text{ and } N$$

Conclusions

- In univalent foundations, SIP can be proved for various set-level mathematical structures
- SIP only holds for *univalent* categories
- Pseudo-conjecture inspired by the case of categories: SIP holds for higher-categorical "univalent" structures

Conclusions

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Thanks for your attention!