Univalent Foundations

II. Univalent type theory

Benedikt Ahrens

Outline

1. Univalent foundations: an interpretation of type theory in simplicial sets
2. Contractible types, equivalences, function extensionality
3. Logic in univalent type theory
4. Homotopy levels
5. Universes and the Univalence Axiom

Moving from classical foundations to univalent foundations

- Mathematics is the study of structures on sets and their higher analogs.
- Set-theoretic mathematics constitutes a subset of the mathematics that can be expressed in univalent foundations.
- Classical mathematics is a subset of univalent mathematics consisting of the results that require LEM and/or AC among their assumptions.

see Voevodsky, Talk at HLF, Sept 2016

Outline

1. Univalent foundations: an interpretation of type theory in simplicial sets
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3. Logic in univalent type theory
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Interpretation of identities as paths

Inhabitants of $\mathsf{Id}(a, a')$ behave like equalities in many ways

- reflexivity, symmetry, transitivity
- $\mathsf{transport}^B : B(x) \times \mathsf{Id}(x, y) \to B(y)$

Inhabitants of $\mathsf{Id}(a, a')$ behave **un**like equalities in one way

Given two identities $p, q : \mathsf{Id}(x, y)$, cannot construct

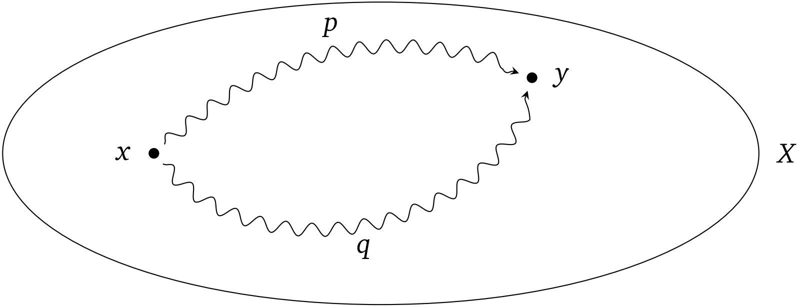
$$\mathsf{Id}_{\mathsf{Id}(x,y)}(p, q)$$

Lack of uniqueness motivates interpretation of elements of $\mathsf{Id}_X(x, y)$ as **paths from** x **to** y **in** X.
A new notation, to reflect the new interpretation:

$$x \rightsquigarrow_X y$$

Identities interpreted as paths in a space

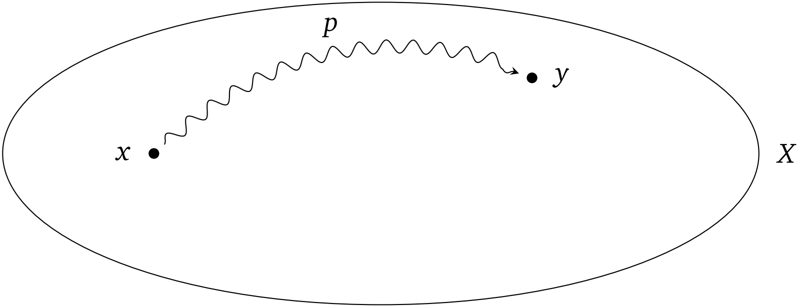
$p, q : x \rightsquigarrow_X y$



'Reflexivity' interpreted as the constant path on a point x.

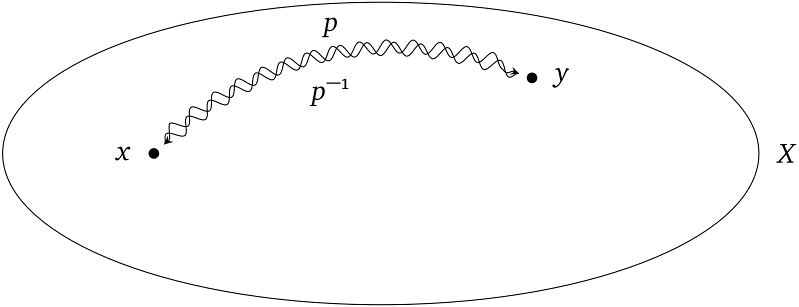
Operations on paths

- $p : x \rightsquigarrow y$



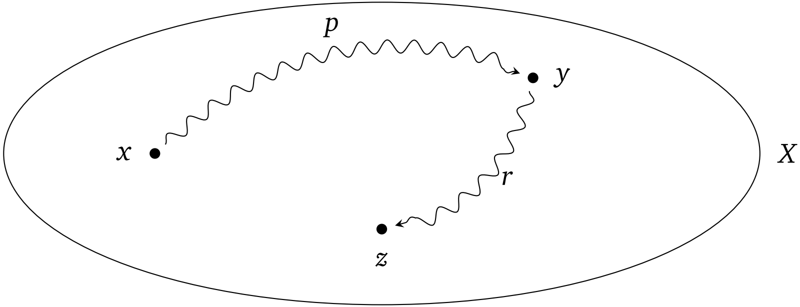
Operations on paths

- $p : x \rightsquigarrow y$
- $p^{-1} : y \rightsquigarrow x$



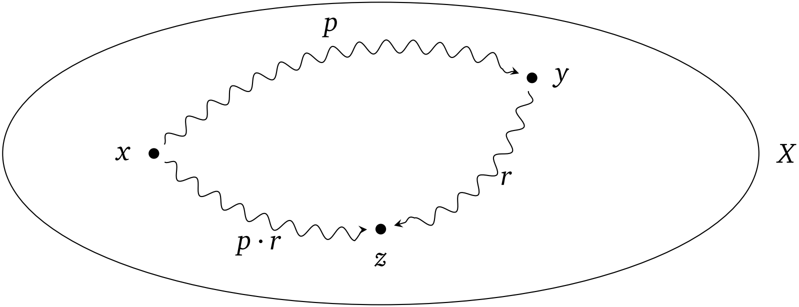
Operations on paths

- $p : x \rightsquigarrow y$
- $p^{-1} : y \rightsquigarrow x$
- $r : y \rightsquigarrow z$



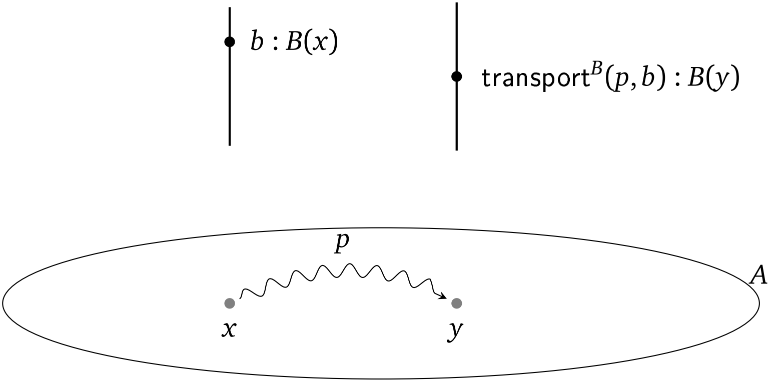
Operations on paths

- $p : x \rightsquigarrow y$
- $p^{-1} : y \rightsquigarrow x$
- $r : y \rightsquigarrow z$
- $p \cdot r : x \rightsquigarrow z$



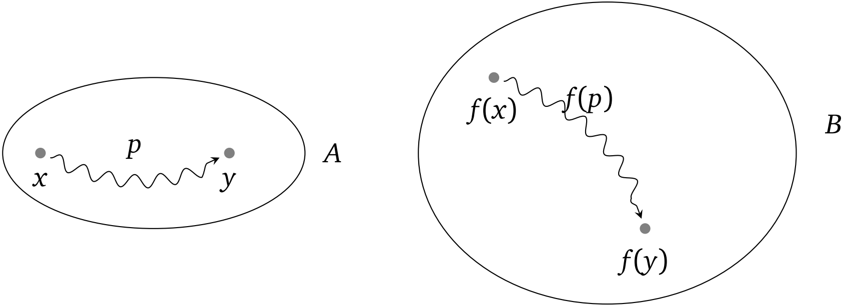
Transport in pictures

$$\mathsf{transport}^B : x \rightsquigarrow y \to B(x) \to B(y)$$



Functions map paths, not just points

$f : A \to B$



Exercise

Given $f : A \to B$, construct a term of type

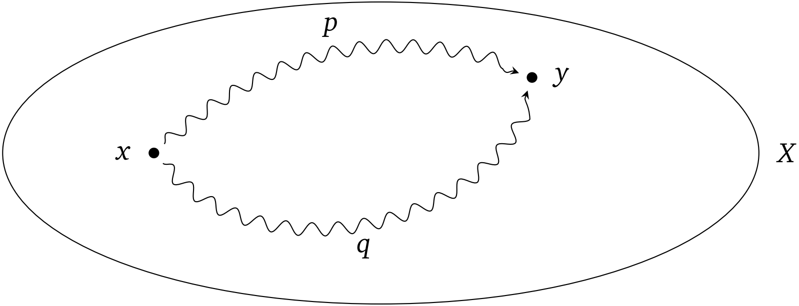
$$\prod_{x,y:A} x \rightsquigarrow_A y \to f(x) \rightsquigarrow_B f(y)$$

Paths between paths

What is a path

$$h : p \rightsquigarrow_{x \rightsquigarrow y} q$$

between paths?



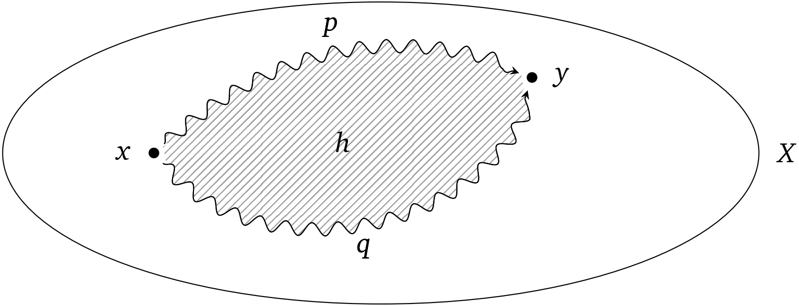
Paths between paths

What is a path

$$h : p \rightsquigarrow_{x \rightsquigarrow y} q$$

between paths?

Intuition: continuous deformation of the first into the second path, called a **homotopy**



Laws satisfied by path concatenation

Can construct homotopies

- $(p \cdot q) \cdot r \rightsquigarrow p \cdot (q \cdot r)$
- $p \cdot 1_y \rightsquigarrow p$
- $1_x \cdot p \rightsquigarrow p$
- $p \cdot p^{-1} \rightsquigarrow 1_x$
- $p^{-1} \cdot p \rightsquigarrow 1_y$

Theorem (Garner, van den Berg)

$$(A, \rightsquigarrow_A, \rightsquigarrow_{\rightsquigarrow_A}, \ldots)$$

forms ∞-groupoid, i.e., groupoid laws hold up to “higher” paths

Interpreting types as topological spaces?

We have not mentioned yet what a “space” or ∞-groupoid is.

Types as topological spaces?

It seems difficult (impossible?) to give a formal interpretation of type theory in the category of topological spaces.

Types as Kan complexes

Vladimir Voevodsky has given an interpretation of type theory in the category of Kan complexes.

There is a ‘Quillen equivalence’ between that category and the category of topological spaces, justifying the intuition of ‘types as (topological) spaces’.

Interpreting types as simplicial sets

Syntax	Simpl. set interpretation
$(A, \rightsquigarrow_A, \rightsquigarrow_{\rightsquigarrow_A}, \ldots)$	Kan complex A
$a : A$	$a \in A_0$
$A \times B$	binary product
$A \to B$	space of maps
$A + B$	binary coproduct
$x : A \ \vdash \ B(x)$	fibration $B \to A$ with fibers $B(x)$
$\sum_{x:A} B(x)$	total space of fibration $B \to A$
$\prod_{x:A} B(x)$	space of sections of fibration $B \to A$

Outline

Contractible types

Definition

The type A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \; :\equiv \; \sum_{x:A} \prod_{y:A} y \rightsquigarrow x$$

A contractible type...

- is also called **singleton** type.
- has a point and a path from any point to that point.

By path inversion and concatenation, there is a path between any two points of a contractible type.

Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \;:\equiv\; \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) \rightsquigarrow b\right)$$

The type of equivalences:

$$A \simeq B \;:\equiv\; \sum_{f:A\to B} \mathsf{isequiv}(f)$$

Exercise: Given an equivalence $f : A \simeq B$, define a function $g : B \to A$. Construct paths $f(g(y)) \rightsquigarrow y$ and $g(f(x)) \rightsquigarrow x$.

Exercises

- Show that 1 is contractible.
- Let A be a contractible type. Construct an equivalence $A \simeq 1$.
- Given types A and B, let $f : A \to B$ and $g : B \to A$. Suppose having families of paths $\eta_x : g(f(x)) \rightsquigarrow x$ and $\epsilon_y : f(g(y)) \rightsquigarrow y$. Show that f is an equivalence.

Path types of pairs

Can construct equivalences (i.e., terms of type)

- for $(a, b) : A \times B$,

$$\Big((a,b) \rightsquigarrow (a',b')\Big) \;\simeq\; \Big((a \rightsquigarrow a') \times (b \rightsquigarrow b')\Big)$$

- for $(a, b) : \sum_{a:A} B(a)$,

$$\Big((a,b) \rightsquigarrow (a',b')\Big) \;\simeq\; \sum_{p:a \rightsquigarrow a'} \mathsf{transport}^B(p,b) \rightsquigarrow b'$$

The equivalences map $\mathsf{refl}(a,b)$ to $\big(\mathsf{refl}(a), \mathsf{refl}(b)\big)$.

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The equivalences map $\mathsf{refl}(a, b)$ to $\big(\mathsf{refl}(a), \mathsf{refl}(b)\big)$.
Exercise: Given functions $f, g : A \to B$, complete

$$\big(f \rightsquigarrow g\big) \;\simeq\; ???$$

Path types of function spaces

For $f, g : A \to B$ cannot show

$$\left(f \rightsquigarrow g\right) \;\simeq\; \left(\prod_{a:A} f(a) \rightsquigarrow g(a)\right)$$

Define

$$\begin{aligned}
\mathsf{toPointwisePath} : \prod_{f,g:A\to B} \left(f \rightsquigarrow g\right) &\to \left(\prod_{a:A} f(a) \rightsquigarrow g(a)\right) \\
\mathsf{toPointwisePath}(f, f, \mathsf{refl}(f)) \;&:\equiv\; \lambda a.\mathsf{refl}(f(a))
\end{aligned}$$

Axiom (function extensionality)

$\mathsf{toPointwisePath}(f, g) : \left(f \rightsquigarrow g\right) \;\to\; \left(\prod_{a:A} f(a) \rightsquigarrow g(a)\right)$ is an equivalence for any f, g.

Outline

Some types are propositions

Curry-Howard

- Types are propositions.
- Terms are proofs.

Univalent logic

- **Some** types are propositions.
- Terms **of those types** are proofs.

Definition (Propositions in univalent type theory)

Type A is a **proposition** if

$$\mathsf{isProp}(A) \quad :\equiv \quad \prod_{x,y:A} x \rightsquigarrow y$$

is inhabited.

Examples of propositions

Exercise: show that

- $\mathtt{1}$ is a proposition.
- any contractible type is a proposition.
- $\mathtt{0}$ is a proposition.
- if A and B are propositions, then $A \times B$ is a proposition.
- if B is a proposition, then $A \to B$ is a proposition.

Connectives in univalent logic

Definition

$$\mathsf{Prop} \;:\equiv\; \sum_{X:\mathsf{Type}} \mathsf{isProp}(X)$$

We want logical connectives

$$\begin{aligned}
\top, \bot &: \mathsf{Prop} \\
\vee, \wedge, \Rightarrow &: \mathsf{Prop} \to \mathsf{Prop} \to \mathsf{Prop} \\
\neg &: \mathsf{Prop} \to \mathsf{Prop} \\
\forall_X, \exists_X &: (X \to \mathsf{Prop}) \to \mathsf{Prop} \qquad \text{(binding a variable)}
\end{aligned}$$

Univalent logic

- $\mathbb{1}$ and $\mathbb{0}$ are propositions. Hence

$$\top \;:\equiv\; \mathbb{1} \qquad \bot :\equiv \mathbb{0}$$

- If A and B are propositions, so is $A \times B$. Hence

$$A \wedge B \;:\equiv\; A \times B$$

- If B is a proposition, so is $A \to B$. Hence

$$A \Rightarrow B \;:\equiv\; A \to B$$

- $\mathbb{0}$ is a proposition, hence $A \to \mathbb{0}$ is. Hence

$$\neg A :\equiv\; A \to \mathbb{0}$$

- If $B(a)$ (for any a) are propositions, so is $\prod_{a:A} B(a)$. Hence

$$\forall (a : A), B(a) \;:\equiv\; \prod_{a:A} B(a)$$

$\vee$ and $\exists$ in univalent logic

- Exercise: Find a type T that is a proposition such that $T + T$ is not a proposition.
  Conclusion: can **not** set

$$A \vee B \;\; :\equiv \;\; A + B$$

- $\Sigma_{n:\mathsf{Nat}} \mathsf{isEven}(n)$ is the type of all even natural numbers. It is not a proposition.
  Conclusion: can **not** set

$$\exists (a : A), B(a) \;\; :\equiv \;\; \Sigma_{a:A} B(a)$$

Solution: introduce a type former that makes propositions.

Propositional truncation

Formation If A is a type, then $||A||$ is a type

Introduction If $a : A$, then $\overline{a} : ||A||$

$$p(A) : \prod_{x,y:||A||} x \rightsquigarrow y$$

Elimination If $f : A \to B$ and B is a proposition, then $\overline{f} : ||A|| \to B$

Computation $\overline{f}(\overline{a}) \equiv f(a)$

- $p(A)$ turns $||A||$ into a proposition.
- Intuitively, $||A||$ is empty if A is, and contractible if A has at least one element.

$\vee$ and $\exists$ in univalent logic

- $$A \vee B \quad :\equiv \quad ||A + B||$$

- $$\exists (a : A), B(a) \quad :\equiv \quad ||\Sigma_{a:A} B(a)||$$

For example:

$$\mathsf{isSurjective}(f) :\equiv \prod_{b:B} ||\Sigma_{a:A} f(a) \rightsquigarrow b||$$

Propositional extensionality

We would like to consider two propositions to be equal if they are logically equivalent:

$$\prod_{P,Q:\mathsf{Prop}} (P \rightsquigarrow Q) \simeq (P \longleftrightarrow Q)$$

Propositional extensionality

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Axiom: propositional extensionality

Exercise: state the axiom of propositional extensionality, e.g., analogously to function extensionality. Is the same care needed as when stating function extensionality?

Exercise

Given $f : A \to B$, show that $\mathsf{isequiv}(f)$ is a proposition.

Outline

Contractible types, propositions and sets

- A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \;:\equiv\; \sum_{x:A} \prod_{y:A} y \rightsquigarrow x$$

- A is a **proposition** if $\prod_{x,y:A} x \rightsquigarrow y$ is inhabited

$$\mathsf{isProp}(A) \;:\equiv\; \prod_{x,y:A} x \rightsquigarrow y$$

- A is a **set** if, for any $x, y : A$, the type $x \rightsquigarrow y$ is a proposition

$$\mathsf{isSet}(A) \;:\equiv\; \prod_{x,y:A} \mathsf{isProp}(x \rightsquigarrow y)$$

Contractible types, propositions and sets

- A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \quad :\equiv \quad \sum_{x:A} \prod_{y:A} y \rightsquigarrow x$$

- A is a **proposition** if $\prod_{x,y:A} \mathsf{isContr}(x \rightsquigarrow y)$ is inhabited

$$\mathsf{isProp}(A) \quad :\equiv \quad \prod_{x,y:A} \mathsf{isContr}(x \rightsquigarrow y)$$

- A is a **set** if, for any $x, y : A$, the type $x \rightsquigarrow y$ is a proposition

$$\mathsf{isSet}(A) \quad :\equiv \quad \prod_{x,y:A} \mathsf{isProp}(x \rightsquigarrow y)$$

Exercises

- For a type A, show that $\prod_{x,y:A} \mathsf{isContr}(x \rightsquigarrow y) \leftrightarrow \prod_{x,y:A} x \rightsquigarrow y$.
- Show that Bool is a set. Is it contractible? Is it a proposition?
- Show that Nat is a set. Is it contractible? Is it a proposition?

Homotopy level of a type

Definition

$$\mathsf{isofhlevel} : \mathsf{Nat} \to \mathsf{Type} \to \mathsf{Type}$$

$$\mathsf{isofhlevel}(0)(X) :\equiv \mathsf{isContr}(X)$$

$$\mathsf{isofhlevel}(S(n))(X) :\equiv \prod_{x,y:X} \mathsf{isofhlevel}(n)(x \rightsquigarrow y)$$

Homotopy level of a type

Definition

$$\mathsf{isofhlevel} : \mathsf{Nat} \to \mathsf{Type} \to \mathsf{Prop}$$

$$\mathsf{isofhlevel}(0)(X) :\equiv \mathsf{isContr}(X)$$

$$\mathsf{isofhlevel}(S(n))(X) :\equiv \prod_{x,y:X} \mathsf{isofhlevel}(n)(x \rightsquigarrow y)$$

Exercise: Show that $\mathsf{isofhlevel}(n)(X)$ is a proposition.

Preservation of levels

. . . by type constructors

- If A and B are of level n, then so is $A \times B$.
- If B is of level n, then so is $A \to B$.

- If A and $B(a)$ (for any $a : A$) are of level n, then so is $\sum_{a:A} B(a)$.
- If $B(a)$ (for any $a : A$) are of level n, then so is $\prod_{a:A} B(a)$.

. . . under equivalence of types

If A is of level n and $A \simeq B$ then B is of level n.

Cumulativity

If type A is of h-level n, then it is also of h-level $S(n)$.

Outline

Reminder:

Universes

There is also a type Type that contains all types, i.e., $A : \mathsf{Type}$.

Exercise: What is the path type of two types?

Fill in

$$A \rightsquigarrow_{\mathsf{Type}} B \quad \simeq$$

Universes

Reminder:

Universes

There is also a type Type that contains all types, i.e., $A : \mathsf{Type}$.

Exercise: What is the path type of two types?

Fill in

$$A \rightsquigarrow_{\mathsf{Type}} B \quad \simeq \quad (A \simeq B)$$

The type theory seen so far does not allow to construct such an equivalence.

Voevodsky's Univalence Axiom

Definition

$$\mathsf{idtoequiv} : \prod_{A,B:\mathsf{Type}} (A \rightsquigarrow B) \to (A \simeq B)$$

$$\mathsf{idtoequiv}(A, A, \mathsf{refl}(A)) \quad :\equiv \quad \mathsf{id}_A$$

Univalence Axiom

$$\mathsf{univalence} : \prod_{A,B:\mathsf{Type}} \mathsf{isequiv}(\mathsf{idtoequiv}_{A,B})$$

Note that univalence is stated for a fixed universe Type.

Consequences of the Univalence Axiom

- Function extensionality

$$\prod_{f,g:A\to B} \left(f \rightsquigarrow g\right) \simeq \left(\prod_{a:A} f(a) \rightsquigarrow g(a)\right)$$

- Propositional extensionality:

$$\prod_{P,Q:\mathsf{Prop}} (P \rightsquigarrow Q) \simeq (P \longleftrightarrow Q)$$

- Paths are isomorphisms for sets:

$$\prod_{X,Y:\mathsf{Set}} (X \rightsquigarrow Y) \simeq (X \cong Y)$$

where

$$X \cong Y \quad :\equiv \quad \sum_{f:X\to Y} \sum_{g:Y\to X} \ldots$$

Summary: Univalent Foundations

- Univalent type theory with an interpretation in spaces (precisely: Kan complexes)

Type theory	Interpretation
A type	space A
$a : A$ (term a of type A)	point a in space A
$f : A \to B$	map from A to B
$p : a \rightsquigarrow b$	path (1-morphism) from a to b in A
$\alpha : p \rightsquigarrow_{a \rightsquigarrow b} q$	homotopy from p to q in A

- "World" of **logic** (propositions and proofs) given by Prop
- "World" of **sets** given by Set (cf. later)